GEODESIC NETS

CONSTRUCTION AND EXISTENCE

by

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Submitted in partial fulfillment of the

requirements for Departmental Honors in

the Department of Mathematics

Texas Christian University

Fort Worth, Texas

May 5, 2025

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ABSTRACT

Geodesic nets are types of graphs in Riemannian manifolds where each edge is a geodesic segment. We present an algorithm for constructing approximate geodesic nets connecting any given number of points in the Euclidean plane. One important object used in the construction of geodesic nets is a balanced vertex, where the sum of unit tangent vectors along adjacent edges is zero. We prove the existence of a balanced vertex of a triangle (with three unbalanced vertices) on a general two-dimensional Riemannian surface when all angles measure less than $2\pi/3$, if the length of the sides of the triangle is not too large. This property is a generalization for the existence of the Fermat point of a planar triangle.

ACKNOWLEDGEMENTS

I would like to thank Dr. Ken Richardson for his dedicated mentorship and valuable discussions for over two years. I also would like to thank Dr. Igor Prokhorenkov and Dr. Michael Scherger for agreeing to be a committee member for this project. Finally, the author would like to thank all the faculty members in the Department of Mathematics and the Department of Computer Science at Texas Christian University for guiding me on my undergraduate journey from my first day in college.

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CHAPTER 1. INTRODUCTION

In [1], the authors introduce *geodesic nets* as certain types of graphs embedded in a Riemannian manifold. A geodesic net on a Riemannian manifold M consists of a finite set V of vertices and a finite set E of edges, which are non-constant geodesics between vertices. The set V contains two types of vertices: *balanced* and *unbalanced*. Unbalanced vertices are vertices of degree 1. Each balanced point v has two properties:

1. The sum of all unit tangent vectors in T_vM from v towards all adjacent vertices is 0.

2. By convention, v must have degree ≥ 3.

Also, the authors in [1] require that no edge connects two unbalanced vertices.

A geodesic net is a generalization of a solution for the *Steiner Tree* problem: given a finite set of points, find the graph spanning all vertices with minimum total length. In the case of three given points, the problem is called Fermat-Torricelli problem. In that case, given a triangle ABC on an Euclidean plane such that all its angles measure less than $2\pi/3$, the Steiner tree contains a point, called a Fermat point, which is connected to $A, B,$ and C and has the three surrounding angles measure $2\pi/3$. The Fermat point here satisfies the property of a balanced point of a geodesic net with three vertices.

In 2021, Parsch, one the authors in [1], investigated a property of a geodesic net with three unbalanced vertices in [2, Theorem 1.2]: Each geodesic net with 3 unbalanced vertices on the plane endowed with a Riemannian metric of non-positive curvature has at most one balanced vertex. This theorem gives an upper bound for the number of balanced vertices in a geodesic net with three unbalanced vertices. In [1, Theorem 3.1.2], Parsch proves that a given geodesic net with three unbalanced vertices on a non-positively curved Riemannian $\mathbb{R}^2$ has exactly one balanced vertex.

In this paper, we investigate some conditions of three unbalanced vertices (or a triangle) for the existence of a balanced vertex given on a general Riemannian surface, not only surfaces with non-positive curvature. To be more specific, let M be a two-dimensional surface and $\mathcal{U}$ be a small neighborhood of a point on M that is homeomorphic to an open disk in $\mathbb{R}^2$. We assume that given two points A and B in $\mathcal{U}$, there exists only one geodesic from A to B that lies inside $\mathcal{U}$, and it is the shortest curve connecting those points. Here is our main theorem showing sufficient conditions for the existence of a balanced point on surfaces with Gaussian curvature bounded above by a constant.

Theorem 1.1. *Let M be a Riemannian surface such that its Gaussian curvature is bounded above by $1/R^2$, for $R > 0$. Let triangle ABC on M be given such that its three angles measure less than $2\pi/3$. If the maximum geodesic distance of two points in the domain of the triangle ABC is less than $R\pi/2$, then there exists a balanced point.*

This theorem generalizes the result of the existence of the Fermat point in a triangle with three angles that measure less than $2\pi/3$ on a plane. A direct corollary of Theorem 1.1 is the following.

Corollary 1.2. *On any neighborhood with compact closure on a complete Riemannian surface, let $R > 0$ be a positive number such that the curvature on that neighborhood is bounded above by $1/R^2$. Thus, if a triangle ABC contained in the neighborhood has three angles that measure less than $2\pi/3$ and the maximum geodesic distance of two points in the domain of the triangle is less than $R\pi/2$, then there exists a balanced point.*

Paper structure: We summarize the structure of the paper as follows:

- In **Chapter 2**, we present the preliminary background, including some definitions, notations, lemmas, and theorems that are used in the proofs of some results later.
- In **Chapter 3**, we show an algorithm constructing geodesic nets with any number of given points on a plane.

- In **Chapter 4**, we prove Theorem 1.1.
  - In **Section 4.1**, we first develop some "warm-up" results and construct the main condition for the existence of a balance point based on the increase of an angle; see Proposition 4.2 and Proposition 4.4.
  - In **Section 4.2**, we prove Proposition 4.6, which is the non-positive-curvature case of the main theorem. This proposition has a simple and direct proof from the "warm-up" results above.
  - In **Section 4.3**, we prove Proposition 4.12 in the case of round spheres and Theorem 1.1 in the case of surfaces with curvature bounded from above. At the end of this subsection, we give one example of a triangle on a round sphere that has no balanced point and one example showing that our condition is not sharp, meaning that there exist triangles on positive-curvature surfaces that have angles greater than $2\pi/3$ and yet have a Fermat point.

Related Results: The definition of a *geodesic net* was first introduced in [3], where Hass and Morgan showed the existence of some specific types of geodesic nets on a 2-sphere. In [4], Heppes extended the previous results by showing that there exists a geodesic net with vertices of degree 3 or 4 partitioning the round 2-sphere into n regions for any natural number n. These two papers were inspired by the famous result stated by Poincaré: There exists a simple geodesic on any positively curved 2-sphere. This result was proved in [5, 6]. In [7], Lyusternik and Fet proved their famous Lyusternik–Fet theorem, which states that there exists a closed geodesic on every compact Riemannian manifold. Another related question is whether every closed manifold has infinitely many periodic geodesics, which is investigated in [8–11]. In [1], Nabutovsky and Parsch survey a wide range of recent problems and results related to geodesic nets, which builds a first step for researchers interested in working with this topic. In 2015, Memarian [12] investigated the problem of the maximum number of balanced vertices in a critical graph, i.e. a planar geodesic net, before the work by Parsch [2]. The connection between minimal networks and geodesic nets on different manifolds is also studied in [13, 14]. Recently, in [15], Chambers et. al. showed that a geodesic flower, which is a finite collection of geodesic loops based at point p that satisfies the balancing condition, is a stationary geodesic net. In [16], Liokumovich and Staffa proved that for a generic Riemannian metric on a closed smooth manifold, the union of the images of all stationary geodesic nets forms a dense set. Following up on Poincaré's question, Dey [17] showed the existence of closed geodesics on certain non-compact Riemannian manifolds.

CHAPTER 2. PRELIMINARY BACKGROUND

In this chapter, we introduce some notations, definitions, lemmas, and theorems that will be used in later results. These definitions are referenced in [18]. Note that our results in this thesis are mainly compatible with the case of smooth Riemannian surfaces, but we will introduce some general definitions on a differentiable manifold M.

2.1 Fundamental definitions in Riemannian geometry

In this section, we give some fundamental definitions of Riemannian geometry that are used in this thesis. First, we present the definitions of differentiable curves, tangent vectors, tangent spaces, and tangent bundles.

Definition 2.1. *Let M be a differentiable manifold. A differentiable function $\alpha : (-\varepsilon, \varepsilon) \to M$, for $\varepsilon > 0$, is called a (differentiable)* ***curve*** *in M. Suppose that $\alpha(0) = p \in M$, and let $\mathcal{D}$ be the set of functions on M that are differentiable at p. Then, the* ***tangent vector to the curve*** *α at $t = 0$ is a function $\alpha'(0) : \mathcal{D} \to \mathbb{R}$ given by*

$$\alpha'(0)f = \left(\frac{d(f \circ \alpha)}{dt}\right)_{t=0}, \quad f \in \mathcal{D}.$$

A ***tangent vector*** *at p is the tangent vector at $t = 0$ of some curve $\alpha : (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$. The set of all tangent vectors to M at p is called the* ***tangent space*** *at p on M, and it is denoted by T_pM. The* ***tangent bundle*** *of M is the set*

$$TM = \{(p, v) | p \in M, v \in T_pM\}.$$

Remark 2.2. *In the case of a smooth Riemannian surface M, T_pM is the tangent plane at a point $p \in M$.*

Next, we define vector fields on M.

Definition 2.3. *A* ***vector field*** *X on a differentiable manifold M is a correspondence that associates to each point $p \in M$ a vector $X(p) \in T_pM$. In terms of mappings, X is a mapping of M into the tangent bundle TM.*

Now, consider a parametrization $\boldsymbol{x} : U \subseteq \mathbb{R}^n \to M$, we can write

$$X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i},$$

where each $a_i : U \to \mathbb{R}$ is a function on U and $\left\{ \frac{\partial}{\partial x_i} \right\}_{i=1}^{n}$ is the basis associates to $\boldsymbol{x}$. Based on this, we can apply the vector field to a differentiable function f on M such that:

$$(Xf)(p) = \sum_{i} a_i(p) \frac{\partial f}{\partial x_i}(p).$$

Next, we have a lemma to define **brackets**.

Lemma 2.4. *Let X and Y be differentiable vector fields on a differentiable manifold M. Then, there exists a unique vector field Z such that, for all $f \in \mathcal{D}$,*

$$Zf = (XY - YX)f.$$

Here, we denote $[X, Y] = Z = XY - YX$ and call it the ***bracket****.*

Next, we define a **Riemannian metric** on a differentiable manifold M.

Definition 2.5. *A* ***Riemannian metric*** *on a differentiable manifold M is a function which associates to each point p of M an inner product $\langle\, ,\, \rangle_p$ (that is, a symmetric, bilinear, positive-definite form) on the tangent space T_pM, which varies differentiably in the following sense: If $\boldsymbol{x} : U \subset \mathbb{R}^n \to M$ is a system of coordinates around p, with $\boldsymbol{x}(x_1, x_2, ..., x_n) = q \in \boldsymbol{x}(U)$ and*

$\left.\frac{\partial}{\partial x_i}\right|_q = d\boldsymbol{x}_q(0, ..., 1, ..., 0)$, *then*

$$\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_q = g_{ij}(x_1, ..., x_n)$$

is a differentiable function on U*, for* $i, j \in \{1, ...n\}$.

Now, we come to the definition of affine connections. Let $\mathcal{X}(M)$ be the set of all vector fields of class C^∞ on M, and let $\mathcal{D}(M)$ be the set of real-valued functions of class C^∞ defined on M.

Definition 2.6. *An* ***affine connection*** ∇ *on a differentiable manifold* M *is a mapping*

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M),$$

which is denoted by $(X, Y) \xrightarrow{\nabla} \nabla_X Y$ *and which satisfies the following properties:*

1. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$.

2. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$.

3. $\nabla_X (fY) = f\nabla_X Y + X(f)Y$,

for $X, Y, Z \in \mathcal{X}(M)$ *and* $f, g \in \mathcal{D}(M)$.

From this definition, we have a proposition that introduces the concepts of **covariant derivative** and **parallel** vector fields.

Proposition 2.7. *Let* M *be a differentiable manifold with an affine connection* ∇*. There exists a unique correspondence which associates a vector field* V *a long a differentiable curve* $c(t) : I \to M$ *to another vector field* $\frac{DV}{dt}$ *along* c*, called the* ***covariant derivative*** *of* V *along* c*, such that: for* W *is a vector field along* c *and* f *is a differentiable function on* I*,*

1. $\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}$.

2. $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$

3. *If V is induced by a vector field $Y \in \mathcal{X}(M)$, i.e., $V(t) = Y(c(t))$, then $\frac{DV}{dt} = \nabla_{dc/dt}Y$.*

Definition 2.8. *Let M be a differentiable manifold with an affine connection ∇. A vector field V along a curve $c : I \to M$ is called **parallel** when*

$$\frac{DV}{dt} = 0, \text{ for all } t \in I.$$

Next, we will introduce the **Levi-Civita** (or **Riemannian**) **connection** on M. First, we present some types of affine connections.

Definition 2.9. *Let M be a differentiable manifold with an affine connection ∇ and a Riemannian metric $\langle \, , \, \rangle$. A connection is said to be **compatible** with the metric $\langle \, , \, \rangle$, when for any smooth curve c and any pair of parallel vector fields X and Y along c, we have $\langle X, Y \rangle$ =constant.*

Definition 2.10. *An affine connection ∇ on a smooth manifold M is said to be **torsion-free** (or **symmetric**) when*

$$\nabla_X Y - \nabla_Y X = [X, Y], \text{ for all } X, Y \in \mathcal{X}(M).$$

Theorem 2.11. *(Levi-Civita) Given a Riemannian manifold M, there exists a unique affine connection ∇ on M satisfying the conditions:*

1. *∇ is torsion-free.*

2. *∇ is compatible with the Riemannian metric.*

The connection given by the theorem is called the **Levi-Civita** (or **Riemannian**) **connection** on M. Next, we introduce one of the main concepts in this paper, which is the term **geodesic**.

Definition 2.12. *A parametrized curve $\gamma : I \to M$ is a **geodesic at** $t_0 \in I$ if $\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0$ at the point t_0. If γ is a geodesic at t, for all $t \in I$, we say γ is a **geodesic**. if $[a, b] \subseteq I$ and $\gamma : I \to M$ is a geodesic, the restriction of γ to $[a, b]$ is called a **geodesic segment joining** $\gamma(a)$ to $\gamma(b)$.*

Remark 2.13. *Consider a geodesic γ in a system of coordinates $(U, \boldsymbol{x})$ about $\gamma(t_0)$. In U, a curve*

$$\gamma(t) = (x_1(t), x_2(t), ..., x_n(t))$$

is a geodesic if and only if

$$0 = \frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = \sum_k \left(\frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma^k_{ij} \frac{dx_i}{dt}\frac{dx_j}{dt}\right)\frac{\partial}{\partial x_k}.$$

Thus, the second-order system of differential equations

$$\frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma^k_{ij} \frac{dx_i}{dt}\frac{dx_j}{dt}, \quad k = 1, ..., n$$

is the system of equations satisfied by every geodesic.

Recall that our main results are on $\mathcal{U}$, a small neighborhood of a point on a two-dimensional surface M, and $\mathcal{U}$ is homeomorphic to an open disk in $\mathbb{R}^2$. From that, we use $\gamma_{A,B}$ to denote the unique geodesic from A to B in $\mathcal{U}$ (assuming the existence and uniqueness of the geodesic). We let $d(A, B)$ be the geodesic distance between A and B; equivalently, it is the length of the shortest curve connecting the two points in M. Next, we have a proposition to define **exponential map**.

Proposition 2.14 (Proposition 2.7 [18]). *Let $p \in M$, there exists a neighborhood V of p in M, a number $\varepsilon > 0$, and a C^∞ mapping $\gamma : (-2, 2) \times \mathcal{V} \to M$, for*

$$\mathcal{V} = \{(q, w) \in TM \mid q \in V, w \in T_q M, |w| < \varepsilon\},$$

such that $t \to \gamma(t, q, w), t \in (-2, 2)$, is the unique geodesic of M which, at $t = 0$, passes through q with velocity w, for every $q \in V$ and for every $w \in T_q M$, with $|w| < \varepsilon$.

Definition 2.15. *Let $p \in M$ and let $\mathcal{V} \subset TM$ be an open set given by Proposition 2.14. Then, the* *map $exp : \mathcal{V} \to M$ given by*

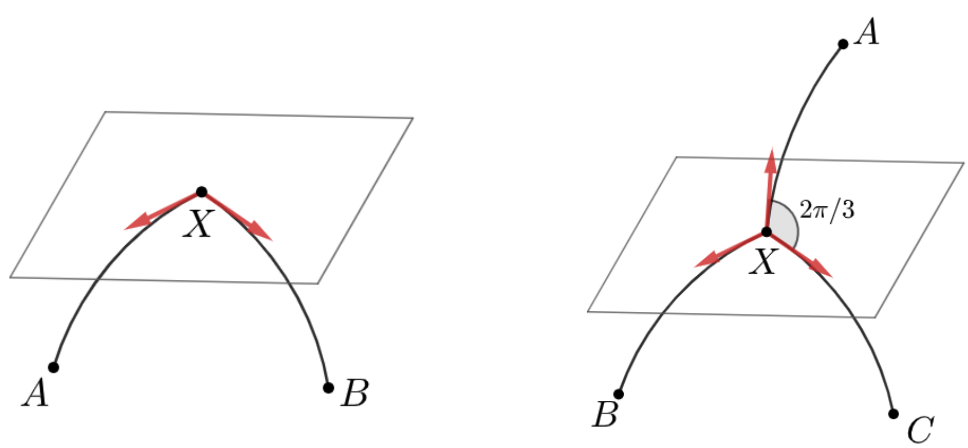
$$exp(q, v) = \gamma(1, q, v) = \gamma\left(|v|, q, \frac{v}{|v|}\right), \quad (q, v) \in \mathcal{V},$$

*is called the **exponential map** on $\mathcal{V}$.*

Lastly, we define an angle between two geodesics and its measure on a Riemannian surface. Recall that our main results work on $\mathcal{U}$, a small neighborhood of a point on a two-dimensional surface M that is homeomorphic to an open disk in $\mathbb{R}^2$. Moreover, for any two points A and B in $\mathcal{U}$, assume that there exists only one geodesic from A to B that lies inside $\mathcal{U}$, and it is the shortest curve connecting those points.

Definition 2.16. *Let A, B, C be three points on $\mathcal{U}$. We define the **angle** between two geodesics $\gamma_{X,A}, \gamma_{X,B}$ at X, denoted by $\angle AXB$, as the smallest angle (among two possible angles) between two tangent vectors $v_{X,A}$ and $v_{X,B}$ of the geodesics $\gamma_{X,A}$ and $\gamma_{X,B}$ at X on the tangent plane $T_X M$, respectively. Also, we denote $m(\angle AXB)$ as the **measure** of $\angle AXB$ with the formula*

$$m(\angle AXB) = \arccos\left(\frac{\langle v_{X,A}, v_{X,B}\rangle}{\|v_{X,A}\|\|v_{X,B}\|}\right).$$



(a) Angle $\angle AXB$ (b) Balanced vertex

Figure 2.1. Angles

Remark 2.17.

- *The definition of the measure of an angle can apply to a higher-dimensional manifold with the same formula based on the inner product.*
- *With this definition, a **balanced vertex** X corresponding to three vertices A, B, C is a point lying inside the triangle ABC (created by geodesics between pairs of vertices) such that*

$$m(\angle AXB) = m(\angle BXC) = m(\angle CXA) = \frac{2\pi}{3}$$

2.2 Jacobi fields

In this section, we will introduce Jacobi fields, which are fundamental objects in Riemannian geometry describing families of geodesics. Jacobi fields will play an important role in the proofs for our main theorem later in this thesis.

Definition 2.18. *Let $\gamma : [0, a] \to M$ be a geodesic in a manifold M. A vector field J along γ is called a **Jacobi field** if it satisfies the Jacobi equation for all $t \in [0, a]$*

$$\frac{D^2 J}{dt^2} + R(\gamma'(t), J(t))\gamma'(t) = 0,$$

where $\frac{DJ}{dt}$ is the covariant derivative of the vector field J along γ, and R is the Riemann curvature operator of M. With the convention from [18],

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z,$$

for X, Y, Z being any three vector fields and ∇ being the Levi-Civita connection.

Lemma 2.19. *In the case of a surface with Gaussian curvature K, if $J(t)$ is orthogonal to $\gamma'(t)$*

for all t, then we can rewrite the Jacobi Equation as

$$\frac{D^2 J}{dt^2} + KJ = 0. \tag{2.1}$$

This follows from **Example 2.3, Chapter 5** in [18]. Now, denote $J'(t) := \frac{DJ}{dt}(t)$. From **Chapter 5** of [18], we have some properties for Jacobi Fields.

Lemma 2.20. *A Jacobi field is determined by its initial conditions $J(0)$ and $J'(0)$.*

Remark 2.21. *If $J(0) = 0$ and $\|J'(0)\| = 0$, then $J \equiv 0$.*

Lemma 2.22. *Let J be a Jacobi field along the geodesic $\gamma : [0, a] \to M$. Then,*

$$\langle J(t), \gamma'(t)\rangle = \langle J'(0), \gamma'(0)\rangle t + \langle J(0), \gamma'(0)\rangle, \quad \textit{for } t \in [0, a].$$

Corollary 2.23. *If we have $J(0) = 0$ and $\langle J'(0), \gamma'(0)\rangle = 0$, then we have $\langle J(t), \gamma'(t)\rangle = 0$ for all $t \in [0, a]$, i.e. the Jacobi fields are perpendicular to the tangent vector $\gamma'(t)$ along the geodesic γ.*

Proposition 2.24. *Let $\gamma : [0, a] \to M$ be a geodesic in M with $\dim M = n$, and let $\mathcal{J}^\perp$ be the space of Jacobi fields with $J(0) = 0$ and $J'(0) \perp \gamma'(0)$. Let $\{J_1(t), \cdots, J_{n-1}(t)\}$ be a basis of $\mathcal{J}^\perp$. If $\gamma(t)$, $t \in (0, a]$, is not conjugate to $\gamma(0)$ (i.e. $J(t) \neq J(0) = 0$), then $\{J_1(t), \cdots, J_{n-1}(t)\}$ is a basis for the orthogonal complement* $\operatorname{span}\{\gamma'(t)\}^\perp \subset T_{\gamma(t)}M$.

Finally, we introduce the **Index Lemma**, a fundamental lemma for proving comparison theorems in differential geometry.

Lemma 2.25 (The Index Lemma [18])**.** *Let $\gamma : [0, a] \to M$ be a geodesic without conjugate points in the interval $(0, a]$. Let J be a Jacobi field along γ, with $\langle J, \gamma'\rangle = 0$, and let V be a piecewise differentiable vector field along γ, with $\langle V, \gamma'\rangle = 0$. Suppose that $J(0) = V(0) = 0$ and that $J(t_0) = V(t_0)$, for $t_0 \in (0, a]$. Then,*

$$I_{t_0}(J, J) \leq I_{t_0}(V, V),$$

where

$$I_{t_0}(V,V) := \int_0^{t_0} (\langle V',V'\rangle - \langle R(\gamma',V)\gamma',V\rangle)dt.$$

Remark 2.26. *In the case of a surface with Gauss curvature K, we have*

$$\begin{aligned} I_{t_0}(V,V) &= \int_0^{t_0} (\langle V',V'\rangle - \langle KV,V\rangle)dt \\ &= \int_0^{t_0} (\langle V',V'\rangle - K\langle V,V\rangle)dt \end{aligned}$$

2.3 Gauss-Bonnet Theorem

In this section, we recall the Gauss-Bonnet Theorem for the case of a triangle and the Lebesgue number lemma. These facts are important for the main proofs below.

Theorem 2.27 (Gauss-Bonnet [19])**.** *Given a smooth complete two-dimensional surface M, let A, B, C be three points on M. Connect points A and B with a directed smooth curve $\alpha_{A,B}$. Let the curves $\alpha_{B,C}$ and $\alpha_{C,A}$ be similarly constructed so that the three curves have empty intersections. We designate a specific domain bounded by three curves as $\mathcal{A}$. Denote the oriented boundary by $\partial\mathcal{A}$. Let $\iota_A, \iota_B, \iota_C$ be three interior angles at three vertices of the triangle. Then, we have*

$$\int_{\mathcal{A}} K\, d\mathcal{A} + \int_{\partial\mathcal{A}} \kappa_g ds = \iota_A + \iota_B + \iota_C - \pi.$$

Corollary 2.28. *If $\alpha_{A,B}, \alpha_{B,C}, \alpha_{C,A}$ are three geodesics, then we obtain*

$$\int_{\mathcal{A}} K\, d\mathcal{A} = \iota_A + \iota_B + \iota_C - \pi.$$

2.4 Lebesgue number

In this section, we introduce the definition of Lebesgue number and the Lebesgue number lemma.

Definition 2.29. *Let $\mathcal{C}$ be an open cover of a metric space X. A* ***Lebesgue number*** *of the cover $\mathcal{C}$*

is a positive real number δ such that for every subset S of X with diameter less than δ, we have

$$S \subseteq C, \text{ for some } C \in \mathcal{C}.$$

Theorem 2.30 (The Lebesgue number lemma [20])**.** *Every open cover of a compact metric space (X, d) has a Lebesgue number.*

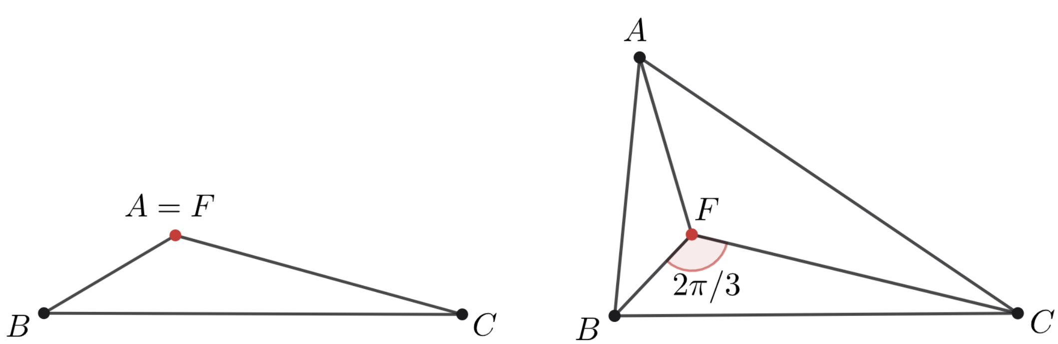
2.5 Fermat point

In this section, we introduce the Fermat point [21] of a triangle on the Euclidean plane. This point plays an important role in our algorithm for constructing geodesic nets on the Euclidean plane.

Definition 2.31. *Given a triangle ABC on the Euclidean plane. The Fermat point F of triangle ABC is the point that minimizes the sum of three distances from three vertices A, B, and C to the point. In other words,*

$$F = \arg\min_X \; \overline{XA} + \overline{XB} + \overline{XC}.$$

Now, we present some geometric properties of the Fermat point in two cases:



(a) Case 1 (b) Case 2

Figure 2.2. Fermat point property in two cases

- **Case 1:** One of three angles of triangle ABC measures greater than or equal to $2\pi/3$. Without loss of generality, assume that $m(\angle BAC) \geq 2\pi/3$. Then, $F = A$.

- **Case 2:** All three angles of triangle ABC measure less than $2\pi/3$. Then, F is located inside the triangle ABC and has the property

$$m(\angle AFB) = m(\angle BFC) = m(\angle CFA) = 2\pi/3.$$

  This property is similar to the "balanced" property of a balanced point with three neighbors in a geodesic net.

CHAPTER 3. CONSTRUCTION ALGORITHM

In this chapter, we present the Bubble Geodesic Net Algorithm (Algorithm 1) for constructing geodesic nets spanning a given set of unbalanced points while minimizing total edge length. The algorithm achieves this by heuristically introducing balanced points and adjusting their positions to satisfy the "balanced" condition. The "balanced" condition in this algorithm requires each balanced point to be connected to exactly three neighbors, with its incident edges forming angles of $2\pi/3$. This angular property ensures local minimality, aligning with the well-known structure of Steiner Trees in the Euclidean plane, which is a network spanning a given set of points with the minimum total length.

3.1 Algorithm Description

3.1.1 Initialization

The algorithm begins with an input set of n unbalanced points, $\{u_1, u_2, \ldots, u_n\}$, and three parameters that affect the balancing process (Subsection 3.1.4):

- m: The number of inner iterations for the balancing process.
- ε: The threshold for convergence in the balancing process.
- α: The tuning parameter for controlling how fast the position of each balanced point changes in every balancing step.

The algorithm maintains two lists:

- *ListUnbalanced*: Stores the given unbalanced points.
- *ListBalanced*: Initially empty, this list stores newly introduced balanced points.

A graph G is constructed by first fully connecting all unbalanced points using the function `FullyConnect(`*`ListUnbalanced`*`)`, which generates a complete graph where edges are weighted by geodesic distances. To simplify the network, the function `MinimumSpanningTree(`G`)` is applied, reducing redundant edges while maintaining connectivity with minimal edge length. Note that there could be many possible Minimum Spanning Trees for the complete graph, and the final geodesic net is determined by the specific output from the `MinimumSpanningTree` function.

3.1.2 Big picture

Here is the big picture of the algorithm. The algorithm checks each unbalanced point v_u:

1. If v_u has degree 1, it remains unchanged.
2. If v_u has a degree greater than 1, the algorithm checks whether any angle formed by its adjacent edges is less than $2\pi/3$.
3. If there is at least one angle in G that is found to be less than $2\pi/3$, the algorithm proceeds to add a new Fermat point and assigns it as balanced point.

If each unbalanced point v_u has degree 1 or connects to two other points with an angle greater than or equal to $2\pi/3$, the algorithm terminates. Otherwise, the process continues by adding a balanced point.

3.1.3 Adding a Balanced Point

When an unbalanced configuration is detected, the algorithm selects a triple (v_1, v_2, v_3) with the smallest internal angle $\widehat{v_1 v_2 v_3}$. A new balanced point v_F is added by using the function `FermatPoint(`v_1, v_2, v_3`)`, which proceeds these steps below:

- Calculate the position of Fermat point v_F
- Add v_F to the set of nodes in G and assigned it as "balanced".

- Create new edges (v_F, v_1), (v_F, v_2), and (v_F, v_3).

- Remove the previous edges (v_2, v_1) and (v_2, v_3).

3.1.4 Balancing process: Adjusting the position of balanced points

In the case when we need to add a new Fermat point of three vertices v_1, v_2, v_3, and at least one of them is balanced, assume that vertex is v_1. Then, after adding the Fermat point v_F, the angles at v_1 may not be equal to $2\pi/3$ since the algorithm changed one edge at v_1. Then, to make v_1 "balanced" again, we need to adjust the positions of all balanced points. The algorithm iteratively adjusts each balanced point v_b based on an update vector Δv, calculated using the function `SumUnitTangent(G,` v_b`)`. The update vector represents the sum of unit tangent vectors along all edges incident to v_b and quantifies local imbalance. The position of v_b is updated iteratively using:

$$v_b = v_b - \alpha \cdot \Delta v, \tag{3.1}$$

where α is the tuning parameter for the update vector. Moreover, convergence is determined by computing a *balance measure*:

$$\mathit{balanceMeasure} = \sum_{v_b \in \mathit{ListBalanced}} \|\Delta v\|. \tag{3.2}$$

If this measure is less than ε, the balancing process terminates. Otherwise, the iterative adjustments continue until the maximum number of iterations m is reached. Note that the choice of the tuning parameter α is important. If α is too large, the updated positions of the balanced points can result in an increasing value of the balance measure.

3.1.5 Algorithm Steps

1. **Initialize:** Store unbalanced points in *ListUnbalanced*, create an empty *ListBalanced*, and construct a fully connected graph G.

2. **Simplify:** Apply `MinimumSpanningTree(`G`)` to reduce redundant edges while maintaining connectivity.

3. **Iterate:**

   - Choose the smallest angle that is less than $2\pi/3$, and add a balanced point.
   - Update the graph by adding the balanced point and restructuring connections.

4. **Balance:** Adjust the positions of balanced points iteratively based on unit tangent vectors until the balance measure is sufficiently small, or until the process reaches the maximum number of iterations.

5. **Terminate:** The algorithm completes when all unbalanced points meet the conditions described in the Subsection 3.1.2

This iterative process ensures that the final geodesic net remains locally minimal, with balanced points maintaining the "balanced" property.

Algorithm 1 Bubble Geodesic Net Algorithm

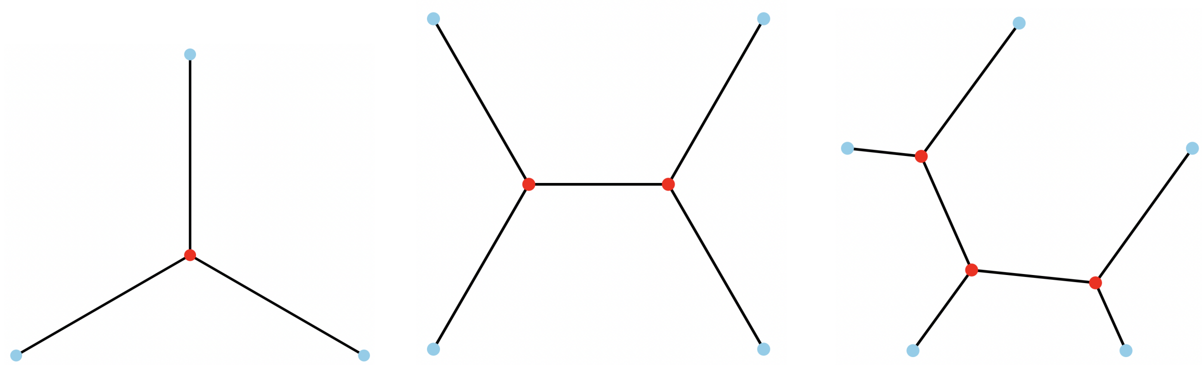
Require: Given n unbalanced points $u_1, u_2, ..., u_n$, number of inner iterations m, measure threshold ε, parameter for update vectors α

Ensure: A geodesic net spanning all unbalanced points

$ListUnbalanced$ = List($u_1, u_2, ..., u_n$)
$ListBalanced$ = List()
G = `FullyConnect`($ListUnbalanced$)
G = `MinimumSpanningTree`(G)
while True **do**
  // Check that all unbalanced point v_u has degree 1 or
  // connect to two points with angle higher or equal to $2\pi/3$
  $unbalanceAngleCount = 0$
  for each node v_u in $ListUnbalanced$ **do**
    if deg(v_u) > 1 **then**
      // If deg(v_u) > 3, then there exists an angle $< 2\pi/3$
      if deg(v_u) > 3 **then**
        $unbalanceAngleCount++$
        break // Break this for loop
      end if
      if the angle at v_u less than $2\pi/3$ **then**
        $unbalanceAngleCount++$
        break // Break this for loop
      end if
    end if
  end for
  if $unbalanceAngleCount == 0$ **then**
    break // Finish the algorithm
  end if
  // Adding Fermat point
  Select the triple (v_1, v_2, v_3) with the smallest angle $\widehat{v_1 v_2 v_3}$
  v_F = `FermatPointReplace`(v_1, v_2, v_3)
  $v_F.isBalanced$ = True
  // Balancing Process
  $balanceMeasure = \infty$
  for $i := 0$ to m **do**
    $balanceMeasure = 0$
    for each node v_b in $ListBalanced$ **do**
      Δv = `SumUnitTangent`(G, v_b)
      $v_b = v_b - \alpha \cdot \Delta v$
      $balanceMeasure = balanceMeasure + \|\Delta v\|$
    end for
    if $balanceMeasure < \varepsilon$ **then**
      break // Finish the balancing process by measure
    end if
  end for
end while

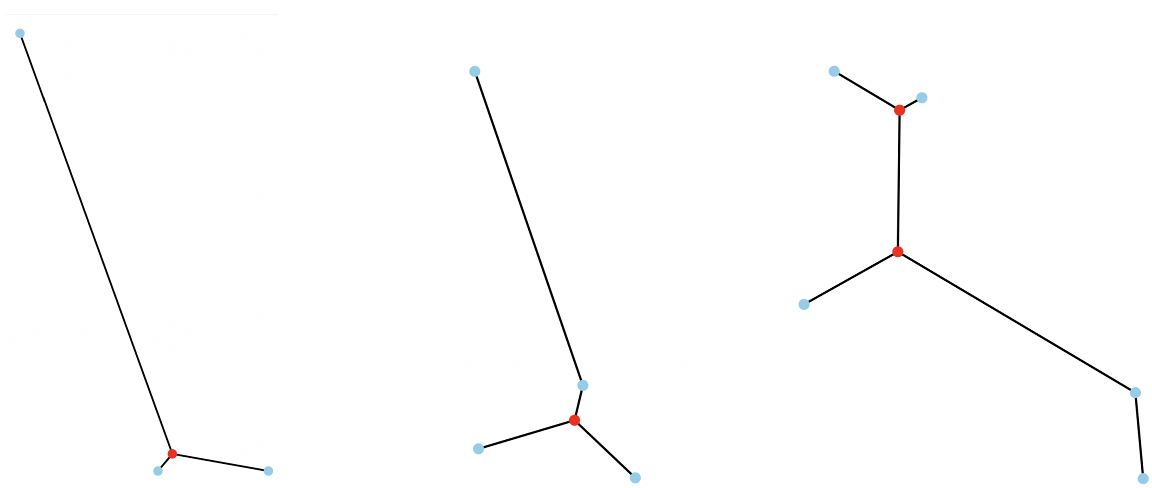
3.2 Implementation and Results

We implement Algorithm 1 at https://github.com/ductoanng/BubbleGeodesicNet. This implementation only constructs geodesic nets on the Euclidean plane. We present the results of applying Algorithm 1 to some sets of unbalanced points. In Figure 3.1, we show the final geodesic nets from three, four, and five initial unbalanced points located at the vertices of the corresponding regular polygons. In Figure 3.2, we show the results for random positions for three, four, and five



(a) 3 unbalanced points (b) 4 unbalanced points (c) 5 unbalanced points

Figure 3.1. Geodesic net constructed with some small numbers of unbalanced points (blue) from vertices of regular polygons. The balanced points are shown in red.

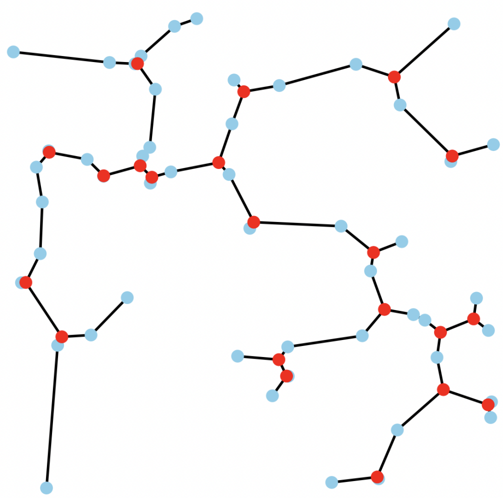


(a) 3 unbalanced points (b) 4 unbalanced points (c) 5 unbalanced points

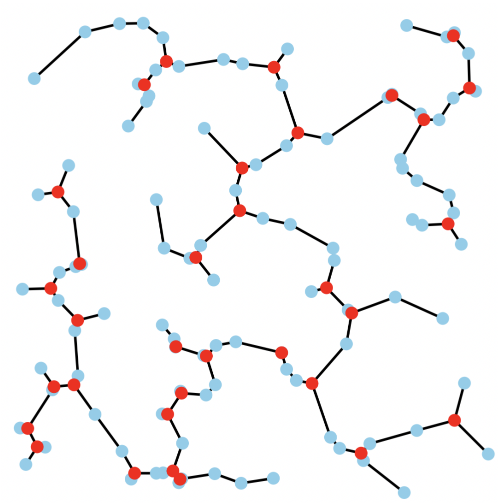
Figure 3.2. Geodesic net constructed with some small numbers of unbalanced points (blue). The balanced points are shown in red.

unbalanced vertices. Next, we increase the given number of unbalanced vertices to 50 and 100.

The locations for these points are generated randomly. We present the final trees of these cases in Figure 3.3.



(a) 50 unbalanced points



(b) 100 unbalanced points

Figure 3.3. Geodesic net constructed with large numbers of unbalanced points (blue). The balanced points are shown in red.

The algorithm efficiently determines the optimal placement of balanced points while maintaining the $2\pi/3$ condition, producing Steiner Trees on the plane that locally minimizes the total edge length.

3.3 Algorithm for general Riemannian surfaces

As seen in the results above, Algorithm 1 performs well on the Euclidean plane. A natural improvement would be to generalize the algorithm to work for arbitrary metrics. However, a key challenge is determining the existence of the Fermat point for three points on a general surface. This challenge motivates us to explore specific conditions for three unbalanced points that ensure the existence of a balanced point on any surface, which we will examine in the next chapter.

CHAPTER 4. EXISTENCE OF A BALANCED VERTEX

On the plane, we recall the existence of a unique balanced vertex, as the Fermat point, of a triangle ABC when the measures all three angles $\angle BAC$, $\angle ACB$, $\angle CBA$ of the triangle are less than $2\pi/3$. In this section, we investigate the existence of a balanced vertex corresponding to three points A, B, C on neighborhood $\mathcal{U}$ on a general surface where all angles of triangle ABC measure less than $2\pi/3$.

4.1 General conditions for the existence of a balanced vertex

Using the notation of the previous section, we first prove the following.

Proposition 4.1. *Let X be a point lying on the geodesic $\gamma_{B,C}$ such that X is different from B and C. Then, there exists a point Y between A and X (exclusively) on $\gamma_{A,X}$ such that $m(\angle BYC) = \dfrac{2\pi}{3}$ (Figure 4.1).*

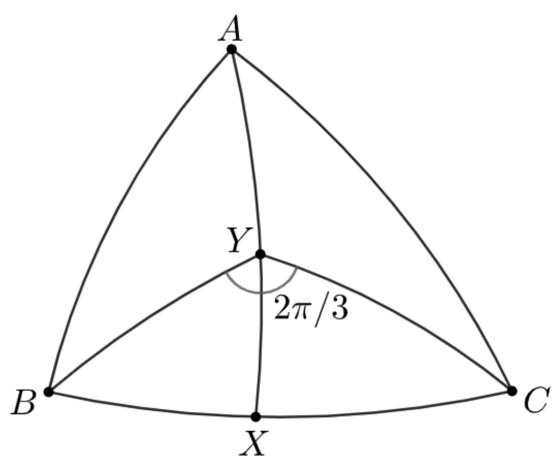


Figure 4.1. Existence of Y with $m(\angle BYC) = \dfrac{2\pi}{3}$ for every X

Proof. Since X is on $\gamma_{B,C}$ and different from B and C, then $m(\angle BXC) = \pi$. Let Y be a moving point on $\gamma_{A,X}$ from A to X (inclusively). When $Y \equiv A$, we have $m(\angle BAC) < \dfrac{2\pi}{3}$; and when $Y \equiv X$, we have $m(\angle BXC) = \pi$. Also, since M is a smooth surface, then Y is moving continuously on $\gamma_{A,X}$ and $m(\angle BYC)$ changes continuously as Y moves. By the Intermediate Value Theorem, there exists a point Y' on $\gamma_{A,X}$ such that $m(\angle BY'C) = \dfrac{2\pi}{3}$. □

From this proposition, for each X on $\gamma_{B,C}$, we denote Y_X as the point that is closest to A and $m(\angle BY_XC) = 2\pi/3$. Note that the existence of the unique point Y_X closest to A can be shown by the continuity of the angle function (i.e. the inverse image of a closed set is closed). Next, we propose a proposition for the existence of a balanced point based on the continuity of Y_X as X moves.

Proposition 4.2. *Let $\triangle ABC$ be a triangle on a neighborhood $\mathcal{U}$ of M such that all angles measure less than $2\pi/3$. Let X be a point lying on the geodesic $\gamma_{B,C}$ such that X is different from B and C. Let Y_X be the closest point to A such that $m(\angle BY_XC) = 2\pi/3$. Assume that the point Y_X continuously varies inside $\triangle ABC$ as a function of X, as X varies on $\gamma_{B,C}$. Then, there exists a point T (balanced point) such that*

$$m(\angle ATB) = m(\angle BTC) = m(\angle CTA) = \frac{2\pi}{3}.$$

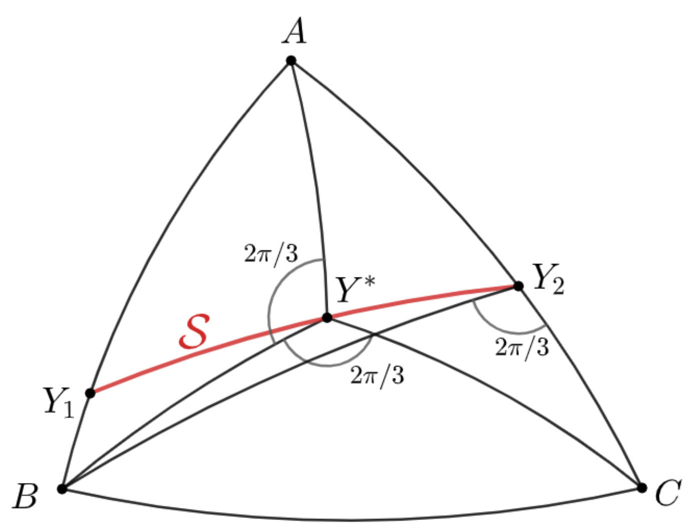


Figure 4.2. Existence of balanced point

Proof. As X moves continuously towards B, since M is a smooth surface, then $\gamma_{A,X}$ also moves continuously towards $\gamma_{A,B}$. Then, since Y_X varies continuously inside $\triangle ABC$, then Y_X converges to a point Y_1 on $\gamma_{A,B}$ (Figure 4.2). Also, since Y_X is always between A and X (exclusively), then Y_1 is between A and B (inclusively). Similarly, as X moves continuously towards C, Y_X moves continuously towards Y_2 between A and C (inclusively) on $\gamma_{A,C}$.

From that, we have Y_X moves on a continuous curve from Y_1 to Y_2 (exclusively), denoted by $\mathcal{S}$. Then, we will prove that there exists a balanced point Y^* on $\mathcal{S}$.

First, we will show that $Y_2 \neq A$ and $m(\angle AY_2B) < \dfrac{2\pi}{3}$.

- **Case 1:** $Y_2 \neq C$. As Y_X moves continuously on M towards Y_2, the measure $m(\angle BY_XC)$ changes continuously. Then,

$$m(\angle BY_2C) = \lim_{X \to C} m(\angle BY_XC) = \lim \frac{2\pi}{3} = \frac{2\pi}{3}.$$

  Since $m(\angle BAC) < \dfrac{2\pi}{3}$, then $Y_2 \neq A$. Also,

$$m(\angle(AY_2B)) = \pi - m(\angle BY_2C) = \pi - \frac{2\pi}{3} = \frac{\pi}{3} < \frac{2\pi}{3}.$$

- **Case 2:** $Y_2 \equiv C$. Then, $Y_2 \equiv C \neq A$. Moreover,

$$m(\angle(AY_2B)) = m(\angle ACB) < \frac{2\pi}{3}.$$

Thus, in both cases, we have $Y_2 \neq A$ and $m(\angle(AY_2B)) < \dfrac{2\pi}{3}$.

Next, we will show that there exists a point Y on $\mathcal{S}$ such that $m(\angle AYB) > 2\pi/3$. From that, by the Intermediate Value Theorem, there exists a balanced point on $\mathcal{S}$. Indeed, consider two cases:

- If $Y_1 \neq B$, then $m(\angle AY_1B) = \pi > \dfrac{2\pi}{3}$. We have Y_X moves continuously on $\mathcal{S}$ from Y_1

to Y_2. In addition, $m(\angle AY_1B) > \frac{2\pi}{3}$, $m(\angle AY_2B) < \frac{2\pi}{3}$, and the measure $m(\angle AY_XB)$ is continuous (due to the smoothness of the manifold). By the Intermediate Value Theorem, there exists Y^* between Y_1 and Y_2 on $\mathcal{S}$ such that $m(\angle AY^*B) = \frac{2\pi}{3}$. Thus, we have

$$m(\angle AY^*B) = m(\angle BY^*C) = m(\angle CY^*A) = \frac{2\pi}{3},$$

whence Y^* is a balanced point.

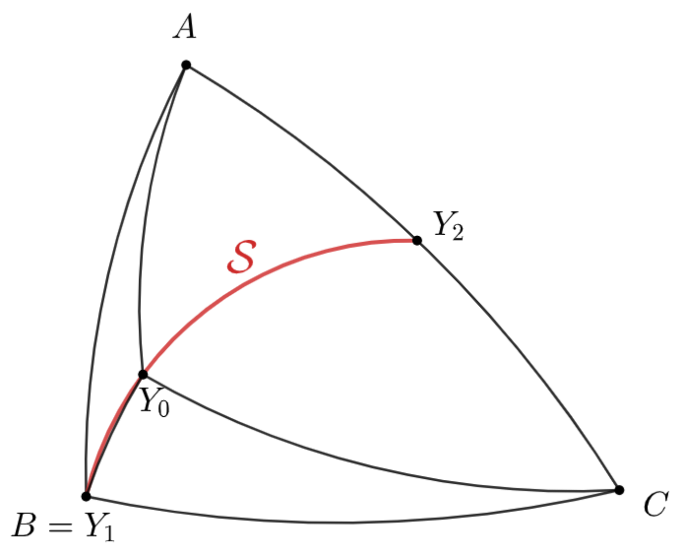


Figure 4.3. Existence of balanced point in case $Y_1 \equiv B$

- If $Y_1 \equiv B$ (Figure 4.3). We will show that the measure of the angle between the curve $\mathcal{S}$ and $\gamma_{B,C}$ at B is $\pi/3$. As Y_X approaches B continuously, we have $\gamma_{Y_X,C}$ approaches $\gamma_{B,C}$, and they are nearly parallel. Let $\epsilon > 0$. Then, there exists Y_X on $\mathcal{S}$ such that

$$m(\angle BY_XC) + m(\angle CBY_X) < \pi + \epsilon,$$

which means

$$m(\angle CBY_X) = \pi + \epsilon - \frac{2\pi}{3} = \frac{\pi}{3} + \epsilon.$$

Thus, as $\epsilon \to 0$, we have the measure of the angle between curve $\mathcal{S}$ and $\gamma_{B,C}$ at B is $\pi/3$.

Moreover, we have $m(\angle ABC) < 2\pi/3$. Denote $\Delta = 2\pi/3 - m(\angle ABC) > 0$. When Y_X tends to $Y_1 \equiv B$, we have $m(\angle Y_X BC)$ tends to $\pi/3$ and $(m(\angle ABY_X)+m(\angle AY_X B))$ tends to π. Let $\varepsilon > 0$ such that $\varepsilon < \Delta$. Then, there exists Y_0 (Figure 4.3) on $\mathcal{S}$ such that

+) $|m(\angle Y_0BC) - \pi/3| < \varepsilon/2$

+) $|m(\angle ABY_0) + m(\angle AY_0B) - \pi| < \varepsilon/2$

Then,

+) $\pi/3 - \varepsilon/2 < m(\angle Y_0BC) < \pi/3 + \varepsilon/2$.

+) $\pi - \varepsilon/2 < m(\angle ABY_0) + m(\angle AY_0B) < \pi + \varepsilon/2$.

Also, we have

$$\begin{aligned} m(\angle ABY_0) &= m(\angle ABC) - m(\angle Y_0BC) \\ &= 2\pi/3 - \Delta - m(\angle Y_0BC) \\ &< 2\pi/3 - \Delta - \pi/3 + \varepsilon/2 \\ &= \pi/3 - \Delta + \varepsilon/2. \end{aligned}$$

Thus,

$$\begin{aligned} m(\angle AY_0B) &> \pi - \varepsilon/2 - m(\angle ABY_0) \\ &> \pi - \varepsilon/2 - (\pi/3 - \Delta + \varepsilon/2) \\ &= 2\pi/3 - \varepsilon + \Delta \\ &> 2\pi/3. \end{aligned}$$

Thus, we have $m(\angle AY_0B) > 2\pi/3$. Similarly to the case $Y_1 \neq B$, by the Intermediate Value Theorem, since $m(\angle AY_0B) > \dfrac{2\pi}{3}$ and $m(\angle(AY_2B)) < \dfrac{2\pi}{3}$, then there exists Y^* between Y_0 and Y_2 on $\mathcal{S}$ such that $m(\angle AY^*B) = \dfrac{2\pi}{3}$. Thus, we also have Y^* is a balanced point.

Therefore, from both cases, we have proved the existence of a balanced point Y^*.

$\square$

Remark 4.3. *In this proposition, the continuity of the points Y_X as a function of $X \in \gamma_{B,C}$ is crucial. This condition will change for different surfaces as the curvature changes.*

To investigate this condition for the continuity of Y_X, we put forward an intermediate proposition.

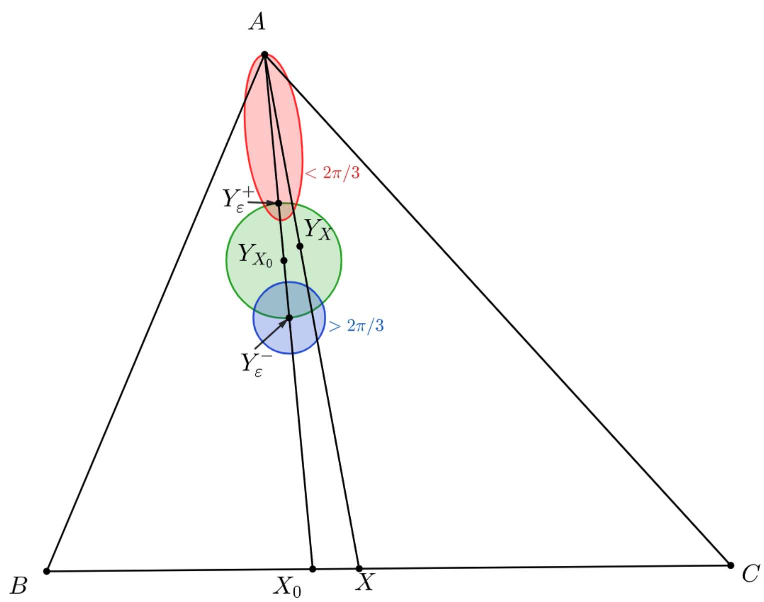
Proposition 4.4. *Given any X on $\gamma_{B,C}$ and Y moving on $\gamma_{A,X}$ from A to X. If the measure $m(\angle BYC)$ is strictly increasing on a neighborhood near Y_X for every X as a function of $d(A, Y_0)$, then the point Y_X is a continuous function of $X \in \gamma_{B,C}$.*

Proof. In this proof, we call the function $m(\angle BYC)$ for $Y \in M$ the angle function. Let X_0 be a fixed point on $\gamma_{B,C}$ between B and C (exclusive), and define the point Y_{X_0} corresponding to X_0. Based on the assumption, there exists a neighborhood around Y_{X_0} such that the measure $m(\angle BYC)$ is strictly increasing as a function of $d(A, Y)$. Thus, there exists a point Y^+ between Y_{X_0} and X_0 such that for all points Y on γ_{A,X_0} between Y_{X_0} and Y^+ (exclusive), we have $m(\angle BYC) > \dfrac{2\pi}{3}$.

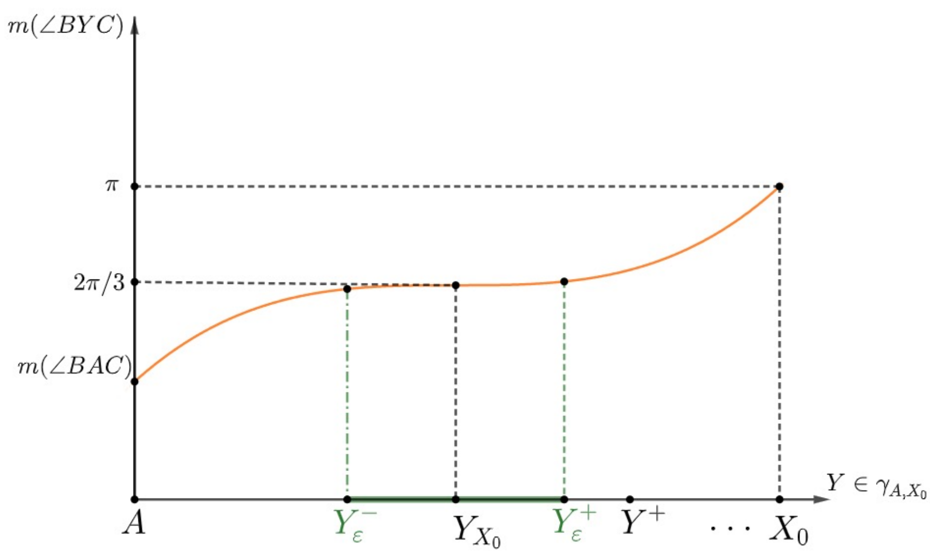
Next, let $\varepsilon > 0$, and assume that $\varepsilon < d(Y_{X_0}, Y^+)$. Let Y_ε^- be a point on γ_{A,X_0} between A and Y_{X_0} such that $d(Y_\varepsilon^-, Y_{X_0}) = \varepsilon$ (Figure 4.4a). Similarly define the point Y_ε^+ between Y_{X_0} and Y^+ such that $d(Y_\varepsilon^+, Y_{X_0}) = \varepsilon$ (Figure 4.4a). To have a better illustration for all points, we can see the graph for the angle function $m(\angle BYC)$ (vertical axis) with respect to the position of Y on γ_{A,X_0}(horizontal axis) in Figure 4.4b.

We will show that there exists a distance δ_ε such that for all points X on $\gamma_{B,C}$ such that $0 < d(X, X_0) < \delta_\varepsilon$, we have the corresponding Y_X is inside the disk

$$B_\varepsilon(Y_{X_0}) = \{Y \in M \mid d(Y, Y_{X_0}) < \varepsilon\}$$



(a) Neighborhoods around Y_{X_0}



(b) Graph for the angle function of $Y \in \gamma_{A,X_0}$

Figure 4.4. Condition for continuity of Y_X

First, notice that the angle function $m(\angle BYC)$ as Y moves on the geodesic $\gamma_{A,Y_\varepsilon^-}$ is a continuous and closed map. Also, the geodesic $\gamma_{A,Y_\varepsilon^-}$ is also a compact set. Thus, the image of the angle function is also compact. By the Extreme Value Theorem, there exists the maximum $\alpha = m(\angle BY^*C)$ for $Y^* \in \gamma_{A,Y_\varepsilon^-}$. Also, Y_{X_0} is the first point that $m(\angle BYC) = \dfrac{2\pi}{3}$ on γ_{A,X_0}. Then, for each $Y \in \gamma_{A,Y_\varepsilon^-}$, we have $m(\angle BYC) \leq \alpha < \dfrac{2\pi}{3}$. Next, since the surface M is smooth everywhere and the function $m(\angle BYC)$ is also continuous as a function of Y, then for each $Y \in \gamma_{A,Y_\varepsilon^-}$, there exists a $\delta_Y^- > 0$ such that the angle function value is less than $\dfrac{2\pi}{3}$ in the neighborhood $B_{\delta_Y^-}(Y)$. Let $\mathcal{A}^-$ be the family of open disks $B_{\delta_Y^-}(Y)$, for all $Y \in \gamma_{A,Y_\varepsilon^-}$. Then, $\mathcal{A}^-$ is an open covering of $\gamma_{A,Y_\varepsilon^-}$, which is a compact set. By the Lebesgue number lemma, $\mathcal{A}^-$ has a Lebesgue number δ^-. Thus, for every $Y \in \gamma_{A,Y_\varepsilon^-}$, the angle function value is less than $\frac{2\pi}{3}$ in the open disk $B_{\frac{\delta^-}{2}}(Y)$ (i.e. the open disk with diameter δ^-). From that, we take the union of all $B_{\frac{\delta^-}{2}}(Y)$ for all $Y \in \gamma_{A,Y_\varepsilon^-}$ to make a **red area** $\mathcal{R}$ in Figure 4.4a. From that, we have the area $\mathcal{R}$ where every point there has the angle function value less than $\dfrac{2\pi}{3}$.

Now, we construct an area around Y_ε^+ such that the angle function value is greater than $\dfrac{2\pi}{3}$. Since the surface is smooth and the angle function is continuous, and $Y_\varepsilon^+ \widehat{B, Y_\varepsilon^+} C > \dfrac{2\pi}{3}$, then there exists $\delta^+ > 0$ such that for each $Y \in B_{\delta^+}(Y_\varepsilon^+)$, we have $m(\angle BYC) > \dfrac{2\pi}{3}$. Then, we have $B_{\delta^+}(Y_\varepsilon^+)$ is the **blue area** $\mathcal{B}$ in Figure 4.4a.

Then, for all X on $\gamma_{B,C}$ such that the geodesic $\gamma_{A,X}$ is totally inside the union of $\mathcal{R}$, $\mathcal{B}$, and $B_\varepsilon(Y_{X_0})$, we have the point Y_X is in $B_\varepsilon(Y_{X_0})$, due to the Intermediate Value Theorem (Figure 4.4a). □

Remark 4.5. *This proposition provides a condition for the continuity of Y_X as of function of X on $\gamma_{B,C}$, which is the strict increase of $m(\angle BYC)$ near Y_X as a function of $d(A, Y)$.*

In the next sections, we will investigate this condition for the continuity of Y_X in two cases: on non-positive curvature surfaces and on arbitrary surfaces with curvature bounded from above.

4.2 Existence of a balanced point on a surface with non-positive curvature

Next, we will show that in the case of a non-positive-curvature surface M. The condition from Proposition 4.4 is satisfied on any triangle on M with no condition on the length of geodesics. Although this result is ultimately a corollary of Theorem 1.1, we include it because of its simplicity.

Proposition 4.6. *Let M be a non-positive-curvature surface. Then, for any triangle on M such that its three angles measure less than $2\pi/3$, there exists a balanced point.*

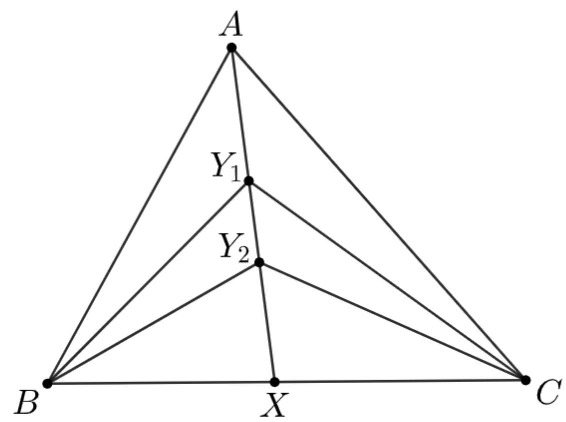


Figure 4.5. Existence of a balanced point in non-positive-curvature case

Proof. Denote $\mathcal{A}$ by the domain of a triangle ABC in M. By the Gauss-Bonnet Theorem, we have

$$\int_{\mathcal{A}} K d\mathcal{A} = m(\angle BAC) + m(\angle CBA) + m(\angle ACB) - \pi.$$

For a non-positive curvature surface, we have $K \leq 0$ everywhere. Then, $\int_{\mathcal{A}} K d\mathcal{A} \leq 0$ and

$$m(\angle BAC) + m(\angle CBA) + m(\angle ACB) \leq \pi.$$

Let $X \in \gamma_{B,C}$, and let $Y_1, Y_2 \in \gamma_{A,X}$ such that Y_1 is between A and Y_2 and $Y_1 \neq Y_2$ (Figure 4.5). For triangle Y_1Y_2B,

$$m(\angle BY_1Y_2) + m(\angle Y_2BY_1) + m(\angle Y_1Y_2B) \leq \pi.$$

Then,

$$\begin{aligned} m(\angle BY_1X) &< m(\angle BY_1X) + m(\angle Y_2BY_1) \quad (m(\angle Y_2BY_1) > 0) \\ &\leq \pi - m(\angle Y_1Y_2B) \\ &= m(\angle BY_2X) \end{aligned}$$

Similarly, we have $m(\angle XY_1C) < m(\angle XY_2C)$. Thus,

$$m(\angle BY_1C) = m(\angle BY_1X) + m(\angle XY_1C) < m(\angle BY_2X) + m(\angle XY_2C) = m(\angle BY_2C).$$

From that, we have the measure $m(\angle BYC)$ increases when Y moves from A to X on $\gamma_{A,X}$, for all $X \in \gamma_{B,C}$. Then, from Proposition 4.4, there exists a balanced point in $\triangle ABC$. □

4.3 Existence of a balanced point on a general surface

In the case of a non-positive curvature surface, the Gauss-Bonnet Theorem implies the sum of three angles in a triangle is less than or equal to π. However, in the case of arbitrary curvature, the sum of three angles in a triangle can be higher than π, which leads to complicated behavior of the angle $\angle BYC$ as Y moves on $\gamma_{A,X}$. In this section, we first investigate the condition from Proposition 4.4 on a general 2-D sphere with radius R. Then, we will extend that condition for a general surface with curvature bounded above by $1/R^2$, which is the curvature of a sphere with radius R. The idea is to use Jacobi fields.

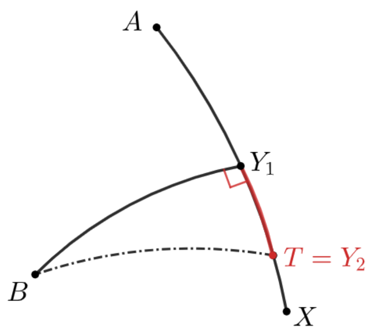
We first come to the proposition of putting a condition for the increase of angles along the geodesic as in Proposition 4.4.

Proposition 4.7. *Let X be a point on $\gamma_{B,C}$. Let Y_1 be a point on geodesic $\gamma_{A,X}$ between A and X. Let Y_2 be a point on $\gamma_{A,X}$ between Y_1 and X, and let T be a point on geodesic γ_{B,Y_2} such that $\gamma_{Y_1,T}$ is orthogonal to γ_{B,Y_1}. Let $\varepsilon > 0$. Assume that Y_2 and T are in a small neighborhood of Y_1 such that*

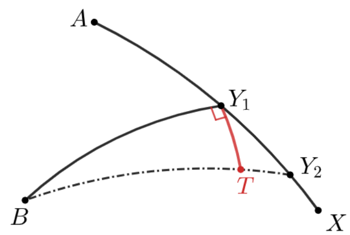
$$\pi - \varepsilon < m(\angle TY_1Y_2) + m(\angle TY_2Y_1) + m(\angle Y_1TY_2) < \pi + \varepsilon. \tag{4.1}$$

Then, we have a condition:

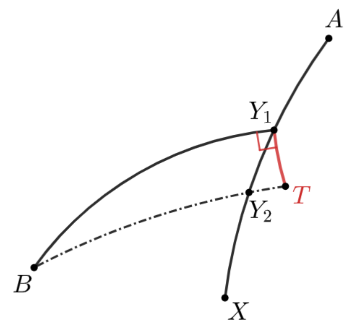
$$\textit{If } m(\angle Y_1TB) < \pi/2 - \varepsilon, \textit{ then } m(\angle BY_2X) > m(\angle BY_1X).$$



(a) Case 1: $m(\angle BY_1X) = \pi/2$



(b) Case 2: $m(\angle BY_1X) > \pi/2$



(c) Case 2: $m(\angle BY_1X) < \pi/2$

Figure 4.6. The increase of angle

Proof. Assume that the angle $\angle Y_1TB$ measures less than $\pi/2$. There are three main cases:

- **Case 1:** $m(\angle BY_1X) = \pi/2$. This case is clear because $T \equiv Y_2$ and

$$m(\angle BY_2X) = \pi - m(\angle BTY_1) > \pi - (\pi/2 - \varepsilon) = \pi/2 + \varepsilon > \pi/2 = m(\angle BY_1X).$$

- **Case 2:** $m(\angle BY_1X) > \pi/2$. Since $\angle BY_1Y_2$ measures larger than $\angle BY_1T$, then the geodesic $\gamma_{Y_1,T}$ is between geodesics $\gamma_{Y_1,B}$ and $\gamma_{Y_1Y_2}$. Thus,

$$\begin{aligned}
m(\angle BY_2X) &= \pi - m(\angle BY_2Y_1) \\
&> \pi + m(\angle TY_1Y_2) + m(\angle Y_2TY_1) - \pi - \varepsilon \quad \text{(From (4.1))} \\
&= m(\angle TY_1Y_2) + (\pi - m(\angle Y_1TB)) - \varepsilon \\
&> m(\angle TY_1Y_2) + \pi - (\pi/2 - \varepsilon) - \varepsilon \\
&= m(\angle TY_1Y_2) + \pi/2 \\
&= m(\angle TY_1Y_2) + m(\angle BY_1T) \\
&= m(BY_1X).
\end{aligned}$$

- **Case 3:** $m(\angle BY_1X) < \pi/2$. We have the geodesic γ_{Y_1,Y_2} is between geodesic $\gamma_{Y_1,B}$ and geodesic $\gamma_{Y_1,T}$. Thus,

$$\begin{aligned} m(\angle BY_2X) &= m(\angle Y_1Y_2T) \\ &> \pi - \varepsilon - m(\angle TY_1Y_2) - m(\angle Y_1TY_2) \quad \text{(From (4.1))} \\ &> \pi - \varepsilon - m(\angle TY_1Y_2) - (\pi/2 - \varepsilon) \\ &= \pi/2 - m(\angle TY_1Y_2) \\ &= m(\angle BY_1T) - m(\angle TY_1Y_2) \\ &= m(\angle BY_1X). \end{aligned}$$

Therefore, in all three cases, we have $m(\angle BY_2X) > m(\angle BY_1X)$. □

Remark 4.8. *Since the proposition is true for all $\varepsilon > 0$, we can obtain the condition:*

If $m(\angle Y_1TB) < \pi/2$ and T is sufficiently close to Y_1, then $m(\angle BY_2X) > m(\angle BY_1X)$.

This proposition allows us to work with right angles instead of general angles in terms of proving the increase of the measures of $\angle BYX$ and $\angle CYX$ at Y_X in Proposition 4.4. From that, we can take advantage of the Jacobi Fields along geodesic $\gamma_{B,Y}$ that are perpendicular to that geodesic. Next, we discuss how the change of the magnitude of Jacobi fields at the point Y_1 relates to the increase in the measure of $\angle BY_1X$.

Proposition 4.9. *Let $\gamma(\theta, t)$ be a variation of geodesics, based on the Gauss lemma (Lemma 3.5 [18]), for some $\epsilon > 0$, given by*

$$\gamma(\theta, t) = \exp_B tv(\theta), \quad 0 \leq t \leq 1, \quad -\epsilon \leq \theta \leq \epsilon,$$

such that $v(\theta)$ is a curve in T_BM, $\gamma(0,t) = \gamma_{B,Y_1}$, $\gamma(0,0) = B$, and $\gamma(0,t_0) = Y_1$, for $t_0 \leq 1$. Assume that $\gamma(\theta, 0) = B$, which means the geodesic is fixed at a boundary point B when θ varies. Let $J(t)$ be a Jacobi field along $\gamma(0,t)$ such that $J(t) = \dfrac{d\gamma}{d\theta}(0,t)$, $J(0) = 0$, $\|J'(0)\| > 0$, and

$\langle J(t), \gamma_t(0,t)\rangle = 0$, *for* $t \in [0, t_0]$. *Assume that* $T = \gamma(\theta_0, t_0)$, *for some* $\theta_0 \in (0, \varepsilon]$, *and* $v(\theta)$ *varies such that* $\gamma(\theta, t_0)$ *is a constant-speed geodesic with parameter* θ. *Let* $f : [0, \theta_0] \to M$ *such that* $f(\theta) = \gamma(\theta, t_0)$ *for* $\theta \in [0, \theta_0]$. *We have a new condition:*

$$\textit{If } (\|J\|^2)'(t_0) > 0, \textit{ then } m(\angle Y_1 T B) < \pi/2.$$

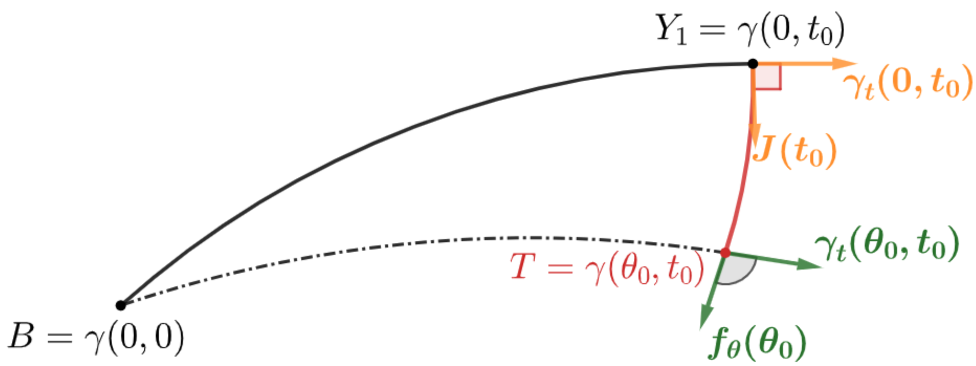


Figure 4.7. Jacobi Field and angles

Proof. Assume that $(\|J\|^2)'(t_0) > 0$. Since $\gamma_{Y_1,T}$ is perpendicular to γ_{B,Y_1}, then $f_\theta(0) \perp \gamma_t(0, t_0)$. Since $J(t_0) \perp \gamma_t(0, t_0)$ and $J(t_0) \in T_{Y_1}M$, then $f_\theta(0) = cJ(t_0)$, for a non-zero scalar c. By construction, $c > 0$.

Next, we have the measure of $\angle Y_1 T B$ is equal to the measure of the angle between $\gamma_t(\theta_0, t_0)$ and $f_\theta(\theta_0)$ (two tangent vectors at T). Then, we have

$$\begin{aligned} m(\angle Y_1 T B) < \pi/2 &\Leftrightarrow \text{The angle between } \gamma_t(\theta_0, t_0) \text{ and } f_\theta(\theta_0) \text{ measures less than } \pi/2 \\ &\Leftrightarrow \langle \gamma_t(\theta_0, t_0), f_\theta(\theta_0)\rangle > 0. \end{aligned}$$

Now, consider the function $g(\theta) = \langle \gamma_t(\theta, t_0), f_\theta(\theta)\rangle$. We have $g(0) = \langle \gamma_t(0, t_0), f_\theta(0)\rangle = 0$. To prove that $g(\theta_0) = \langle \gamma_t(\theta_0, t_0), f_\theta(\theta_0)\rangle > 0$, we will show that $\left(\dfrac{dg}{d\theta}\right)_{\theta=0} > 0$.

Indeed, we have

$$\left(\frac{dg}{d\theta}\right)_{\theta=0} = \left(\frac{d}{d\theta}\langle\gamma_t(\theta,t_0), f_\theta(\theta)\rangle\right)_{\theta=0}$$
$$= \left\langle \left(\frac{D}{\partial\theta}\gamma_t\right)_{(\theta,t)=(0,t_0)}, f_\theta(0)\right\rangle + \left\langle \gamma_t(0,t_0), \left(\frac{D}{d\theta}f_\theta(\theta)\right)_{\theta=0}\right\rangle$$

We have $f(\theta)$ parametrizes a geodesic, then $\frac{D}{d\theta}f_\theta(\theta) = 0$, for $\theta \in [0,\theta_0]$. Now we want to show that

$$\frac{D}{\partial\theta}\gamma_t = \frac{D}{\partial t}\gamma_\theta$$

Indeed, let ∂_θ and ∂_t be two coordinate vector fields of the surface near $\gamma(0,t_0)$. Let ∇ be the Levi-Civita connection on M. Then,

$$\frac{D}{\partial\theta}\partial_t = \nabla_{\partial_\theta}\partial_t \quad \text{and} \quad \frac{D}{\partial t}\partial_\theta = \nabla_{\partial_t}\partial_\theta,$$

with $\partial_t = \gamma_t(\theta,t)$ and $\partial_\theta = \gamma_\theta(\theta,t)$, which is the same as $f_\theta(\theta)$ when $t = t_0$. Since γ is smooth, then

$$[\partial_\theta,\partial_t] = \partial_\theta\partial_t - \partial_t\partial_\theta = 0.$$

Thus, we have

$$\frac{D}{\partial\theta}\partial_t - \frac{D}{\partial t}\partial_\theta = \nabla_{\partial_\theta}\partial_t - \nabla_{\partial_t}\partial_\theta = [\partial_\theta,\partial_t] = 0.$$

Therefore,

$$\frac{D}{\partial\theta}\gamma_t = \frac{D}{\partial t}\gamma_\theta.$$

Next, we have

$$\begin{aligned}\left\langle \left(\frac{D}{\partial\theta}\gamma_t\right)_{(\theta,t)=(0,t_0)}, f_\theta(0)\right\rangle &= \left\langle \left(\frac{D}{\partial t}\gamma_\theta\right)_{(\theta,t)=(0,t_0)}, f_\theta(0)\right\rangle \\ &= \left\langle \left(\frac{D}{dt}\gamma_\theta(0,t)\right)_{t=t_0}, f_\theta(0)\right\rangle \\ &= \left\langle \left(\frac{D}{dt}J(t)\right)_{t=t_0}, cJ(t_0)\right\rangle \\ &= c\langle J', J\rangle(t_0) \\ &= \frac{c}{2}(\langle J, J\rangle)'(t_0) \\ &= \frac{c}{2}(\|J\|^2)'(t_0) \\ &> 0.\end{aligned}$$

Therefore,

$$\left(\frac{dg}{d\theta}\right)_{\theta=0} = \underbrace{\left\langle \left(\frac{D}{\partial\theta}\gamma_t\right)_{(\theta,t)=(0,t_0)}, f_\theta(0)\right\rangle}_{>0} + \left\langle \gamma_t(0,t_0), \underbrace{\left(\frac{D}{d\theta}f_\theta(\theta)\right)_{\theta=0}}_{=0}\right\rangle > 0.$$

□

By combining this proposition with Proposition 4.7, we just need to investigate when $(\|J\|^2)'(t_0) > 0$ to achieve the increasing-angle condition in Proposition 4.4. This is also the main condition we will use for the next results in the case of surfaces with curvatures bounded from above.

Now, we start with the constant positive curvature case by investigating $(\|J(t)\|^2)'$ on a round sphere with radius R. The below proposition is shown in **Example 2.3, Chapter 5** of [18].

Proposition 4.10. *Let γ be a normalized geodesic on a round sphere with radius R. Let J be a Jacobi field along γ that is orthogonal to γ'. Let $w(t)$ be a parallel field along γ with $\langle\gamma'(t), w(t)\rangle = 0$*

and $\|w(t)\| = 1$. *Then,*

$$J(t) = R\sin\left(\frac{t}{R}\right)w(t).$$

Corollary 4.11. *If γ is a normalized geodesic, i.e. $\|\gamma'\| = 1$, then $(\|J(t)\|^2)' > 0$ for all $t \in (0, R\pi/2)$. Therefore, if the maximum geodesic distance of two points in the domain of the triangle ABC on a sphere with radius R is less than $R\pi/2$, it will satisfy the condition in Proposition 4.4.*

Proof. From Proposition 4.10, we have

$$\|J(t)\|^2 = \langle J(t), J(t)\rangle = R^2\sin^2(t/R).$$

Then,

$$(\|J(t)\|^2)' = (R^2\sin^2(t/R))' = 2R\sin(t/R)\cos(t/R).$$

Thus, for all $t \in (0, R\pi/2)$, we have

$$(\|J(t)\|^2)' = 2R\sin(t/R)\cos(t/R) > 0.$$

□

From this, we can describe the condition for the angle increasing through the Jacobi Field on 2-D spheres. Next, we combine Corollary 4.11 with Proposition 4.4 to prove the existence of a balanced point on a round sphere.

Proposition 4.12. *Given triangle ABC on a round sphere M with Gaussian curvature $1/R^2$ such that its three angles measure less than $2\pi/3$. If the maximum geodesic distance of two points in the domain of the triangle ABC is less than $R\pi/2$, then there exists a balanced point.*

Proof. Let X be a point on $\gamma_{B,C}$ and Y be a point on $\gamma_{A,X}$. We have the maximum lengths of $\gamma_{B,Y}$ and $\gamma_{C,Y}$ are both less than $R\pi/2$. By Corollary 4.11, we have $m(\angle BYX)$ and $m(\angle CYX)$ are increasing as Y moves from A to X (Figure 4.8). From that, by Proposition 4.4, there exists a balanced point in the triangle ABC. □

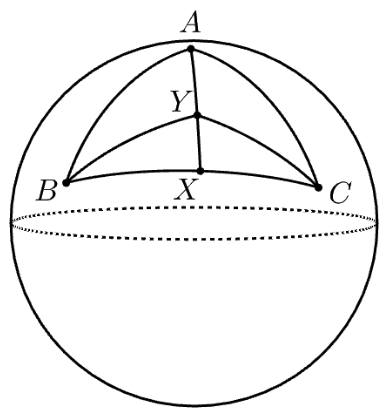


Figure 4.8. Triangle ABC on a sphere

Now, we generalize the condition to general surfaces with Gaussian curvature K bounded above by the curvature of a sphere as Theorem 1.1

Proof of Theorem 1.1. Let X be a point on $\gamma_{B,C}$ and Y be a point on $\gamma_{A,X}$. Let $\gamma(t) : [0, a] \to M$ parameterize $\gamma_{B,X}$ such that $\gamma(0) = B, \gamma(a) = X$, and $\|\gamma'(t)\| = 1$, for $t \in [0, a]$. Let $J(t)$ be a Jacobi field along $\gamma(t)$ such that $J(0) = 0$, $\|J'(0)\| > 0$, and $\langle J(t), \gamma'(t)\rangle = 0$, for all $t \in [0, a]$. We need to prove $(\|J(t)\|^2)' > 0$ so that we can use Proposition 4.9 and Proposition 4.7 to achieve the increase-angle condition in Proposition 4.4.

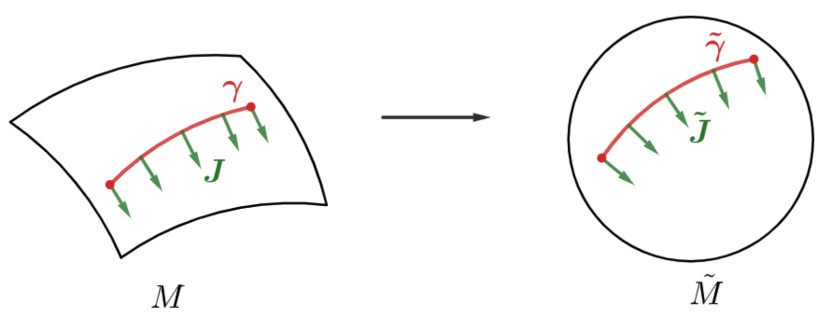


Figure 4.9. Relating Jacobi fields in M and $\widetilde{M}$

Let $\widetilde{M}$ be a sphere with radius R. Then, M has the Gaussian curvature $\widetilde{K} = \frac{1}{R^2}$. Let $\widetilde{\gamma}$: $[0, a] \to \widetilde{M}$ be a geodesic with unit velocity (i.e. $\|\widetilde{\gamma}'(t)\| = 1 = \|\gamma'(t)\|$). Then, $\widetilde{\gamma}(t)$ has the same length as $\gamma(t)$ as $t \in [0, a]$ that is less than $R\pi/2$. Let $\tilde{J}(t)$ be a Jacobi field along $\widetilde{\gamma}$ satisfying these conditions:

- $\tilde{J}(0) = J(0) = 0$

- $\langle \tilde{J}(t), \widetilde{\gamma}'(t) \rangle = \langle J(t), \gamma'(t) \rangle = 0$, for all $t \in [0, a]$

- $\|\tilde{J}'(0)\| = \|J'(0)\| > 0$.

Since the length of $\widetilde{\gamma}$ is less than $R\pi/2$, then by Corollary 4.11, we have $(\|\tilde{J}(t)\|^2)' > 0$, for all $t \in [0, a]$.

Now, let $v(t) := \|J(t)\|^2$ and $\tilde{v}(t) := \|\tilde{J}(t)\|^2$. We have the length of $\widetilde{\gamma}(t)$ for $t \in (0, a]$ is less than $R\pi/2$. Then $\widetilde{\gamma}(t)$ has no conjugate point when $t \in (0, a]$. Thus, $\tilde{v}(t) = \|\tilde{J}(t)\|^2 > 0$, for all $t \in (0, a]$. We also have $\tilde{v}'(t) = (\|\tilde{J}(t)\|^2)' > 0$, for all $t \in [0, a]$. Our goal is showing that $v'(t) > 0$, for all $t \in (0, a]$. First, we show that $v(t) \geq \tilde{v}(t)$, for $t \in (0, a]$. Indeed, from L'Hospital's rule,

$$\begin{aligned}
\lim_{t\to 0^+} \frac{v(t)}{\tilde{v}(t)} &= \lim_{t\to 0^+} \frac{v'(t)}{\tilde{v}'(t)} = \lim_{t\to 0^+} \frac{\langle J(t), J(t)\rangle'}{\langle \tilde{J}(t), \tilde{J}(t)\rangle'} = \lim_{t\to 0^+} \frac{2\langle J'(t), J(t)\rangle}{2\langle \tilde{J}'(t), \tilde{J}(t)\rangle} \\
&= \lim_{t\to 0^+} \frac{\langle J''(t), J(t)\rangle + \langle J'(t), J'(t)\rangle}{\langle \tilde{J}''(t), \tilde{J}\rangle + \langle \tilde{J}'(t), \tilde{J}'(t)\rangle} \\
&= \frac{\langle J''(0), J(0)\rangle + \langle J'(0), J'(0)\rangle}{\langle \tilde{J}''(0), \tilde{J}(0)\rangle + \langle \tilde{J}'(0), \tilde{J}'(0)\rangle} \\
&= \frac{\|J'(0)\|^2}{\|\tilde{J}'(0)\|^2} \qquad \text{(Since } J(0) = \tilde{J}(0) = 0\text{)} \\
&= 1.
\end{aligned}$$

Therefore, to prove that $v(t) \geq \tilde{v}(t)$, we need to show that $\frac{d}{dt}\left(\frac{v(t)}{\tilde{v}(t)}\right) \geq 0$, or equivalently,

$$v'(t)\tilde{v}(t) \geq v(t)\tilde{v}'(t), \quad \text{for all } t \in (0, a]. \tag{4.2}$$

Now, fix $t_0 \in (0, a]$. If $v(t_0) = 0$, then $J(t_0) = 0$ and

$$v'(t_0) = 2\langle J'(t_0), J(t_0)\rangle = 0.$$

Then, both sides of (4.2) are 0.

Suppose that $v(t_0) \neq 0$. Then, define

$$U(t) = \frac{1}{\sqrt{v(t_0)}} J(t), \quad \tilde{U}(t) = \frac{1}{\sqrt{\tilde{v}(t_0)}} \tilde{J}(t).$$

Then,

$$\begin{aligned}
\frac{v'(t_0)}{v(t_0)} &= \frac{2\langle J'(t_0), J(t_0)\rangle}{\langle J(t_0), J(t_0)\rangle} = 2\langle U'(t_0), U(t_0)\rangle = 2\langle U', U\rangle(t_0) \\
&= 2\int_0^{t_0} \langle U', U\rangle' dt = 2\int_0^{t_0} (\langle U', U'\rangle + \langle U'', U\rangle) dt \\
&= 2\int_0^{t_0} (\langle U', U'\rangle - \langle KU, U\rangle) dt \\
&= 2\int_0^{t_0} (\langle U', U'\rangle - K\langle U, U\rangle) dt \\
&= 2I_{t_0}(U, U).
\end{aligned}$$

Similarly, we have

$$\frac{\tilde{v}'(t_0)}{\tilde{v}(t_0)} = 2I_{t_0}(\tilde{U}, \tilde{U}).$$

Therefore, to show $\dfrac{v'(t_0)}{v(t_0)} \geq \dfrac{\tilde{v}'(t_0)}{\tilde{v}(t_0)}$, we need to prove that

$$I_{t_0}(U, U) \geq I_{t_0}(\tilde{U}, \tilde{U}).$$

Indeed, let $\{e_1(t), e_2(t)\}$ and $\{\tilde{e}_1(t), \tilde{e}_2(t)\}$ be parallel orthonormal bases along $\gamma(t)$ and $\widetilde{\gamma}(t)$, respectively, such that

$$e_1(t) = \gamma'(t)/\|\gamma'(t)\|, \quad e_2(t) = U(t_0),$$

$$\tilde{e}_1(t) = \widetilde{\gamma}'(t)/\|\widetilde{\gamma}'(t)\|, \quad \tilde{e}_2(t) = \tilde{U}(t_0).$$

Then, write $U(t) = g_1(t)e_1(t) + g_2(t)e_2(t)$ along γ. Also, define a map

$$\phi : T_{\gamma(t)}M \to T_{\widetilde{\gamma}(t)}M$$

such that

$$(\phi U)(t) = g_1(t)\tilde{e}_1(t) + g_2(t)\tilde{e}_2(t).$$

Then, we have these properties:

$$\begin{aligned}
\langle \phi U, \phi U \rangle &= \langle g_1(t)\tilde{e}_1(t) + g_2(t)\tilde{e}_2(t), g_1(t)\tilde{e}_1(t) + g_2(t)\tilde{e}_2(t) \rangle \\
&= g_1^2(t) + g_2^2(t) \\
&= \langle g_1(t)e_1(t) + g_2(t)e_2(t), g_1(t)e_1(t) + g_2(t)e_2(t) \rangle \\
&= \langle U, U \rangle
\end{aligned}$$

and

$$(\phi U)' = \big(g_1(t)\tilde{e}_1(t) + g_2(t)\tilde{e}_2(t)\big)' = g_1'(t)\tilde{e}_1(t) + g_2'(t)\tilde{e}_2(t) = \phi(U').$$

From that, we also have

$$\langle (\phi U)', (\phi U)' \rangle = \langle \phi(U'), \phi(U') \rangle = \langle U', U' \rangle.$$

Since $K \leq \widetilde{K}$, we have

$$\begin{aligned}
I_{t_0}(U, U) &= \int_0^{t_0} (\langle U', U' \rangle - K \langle U, U \rangle) dt \\
&\geq \int_0^{t_0} (\langle U', U' \rangle - \widetilde{K} \langle U, U \rangle) dt \\
&= \int_0^{t_0} (\langle (\phi U)', (\phi U)' \rangle - \widetilde{K} \langle \phi U, \phi U \rangle) dt \\
&= I_{t_0}(\phi U, \phi U).
\end{aligned}$$

Thus,

$$I_{t_0}(U,U) \geq I_{t_0}(\phi U, \phi U).$$

On the other hand, we have $\tilde{U}$ and ϕU are two vector fields along $\widetilde{\gamma}$ that satisfy all hypothesis conditions in Lemma 2.25. Also, $\tilde{U}$ is a Jacobi field along $\widetilde{\gamma}$. By Lemma 2.25, we have

$$I_{t_0}(\phi U, \phi U) \geq I_{t_0}(\tilde{U}, \tilde{U}).$$

Therefore,

$$I_{t_0}(U,U) \geq I_{t_0}(\phi U, \phi U) \geq I_{t_0}(\tilde{U}, \tilde{U}).$$

Hence, we have $v(t) \geq \tilde{v}(t)$, for all $t \in (0, a]$. Moreover, $\tilde{v}(t) > 0$, then $v(t) \geq \tilde{v}(t) > 0$, for all $t \in (0, a]$. From inequality (4.2), since $v(t), \tilde{v}(t)$, and $\tilde{v}'(t)$ are all positive on $(0, a]$, then $v'(t) > 0$, for all $t \in (0, a]$. Thus, the increase-angle condition in Proposition 4.4 is satisfied. Therefore, there exists a balanced point inside triangle ABC. □

Remark 4.13. *This proof is mostly covered in the proof of The Rauch Comparion Theorem [18]. In this problem, we just consider the case of a 2-D surface instead of an n-dimensional manifold.*

From Proposition 4.12 and Theorem 1.1, the bounded maximum geodesic distance in the region of triangle ABC is a crucial condition. That fact raises a question: *Is there any triangle ABC that has the maximum side length larger or equal to $R\pi/2$ that does not have a balanced point?*

Here, we construct a simple example to show the answer “yes" to the above question.

Example 4.14. *Let M be a 2-D sphere with radius R. Let A and C be two distinct points on the sphere, and let B be the antipodal point of A (Figure 4.10). Then, the triangle ABC does not have a balanced vertex Y inside the triangle because for all points Y inside $\triangle ABC$, the measure of $\angle BYA$ is always π, which cannot be $2\pi/3$.*

In addition, we show an example where the side lengths are greater than $R\pi/2$, and where there still exists a balanced point.

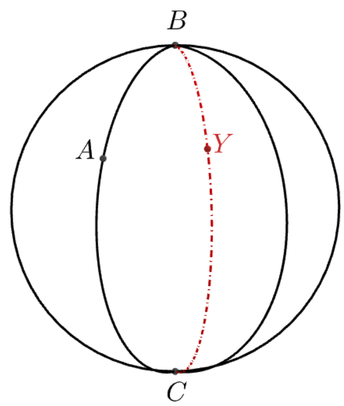


Figure 4.10. Example of a triangle having no balanced point

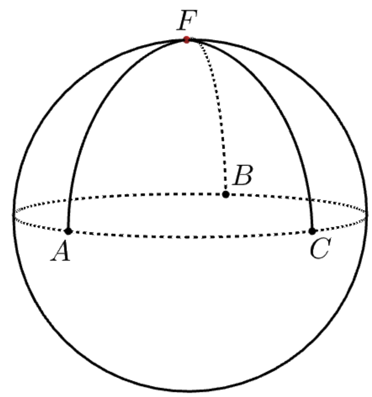


Figure 4.11. Example of a triangle with side lengths larger than $R\pi/2$ which has a balanced point.

Example 4.15. *Let M be a 2-D sphere with radius R. On the horizontal equator, take 3 points A, B, C such that triangle ABC is equilateral. Also, take F as one of two poles of the sphere (Figure 4.11). We have each side length of triangle ABC is $\frac{2R\pi}{3} > \frac{R\pi}{2}$. Also, we have*

$$m(\angle AFB) = m(\angle BFC) = m(\angle CFA) = 2\pi/3.$$

That means F is a balanced point of triangle ABC.

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